Journal of Approximation Theory **109**, 198–217 (2001)
doi:10.1006/jath.2000.3533, available online at http://www.idealibrary.com on IDEAL®

# Strong Asymptotics of the Generating Polynomials of the Stirling Numbers of the Second Kind

Christian Elbert

*Department of Mathematics, University of Trier, 54286 Trier, Germany*
E-mail: christian.elbert@dglux.lu

*Communicated by Doron S. Lubinsky*

Received September 1, 1999; accepted in revised form September 12, 2000;
published online February 16, 2001

For the horizontal generating functions $P_n(z)=\sum_{k=1}^{n} S(n,k)\, z^k$ of the Stirling numbers of the second kind, strong asymptotics are established, as $n \to \infty$. By using the saddle point method for $Q_n(z)=P_n(nz)$ there are two main results: an oscillating asymptotic for $z \in (-e, 0)$ and a uniform asymptotic on every compact subset of $\mathbb{C} \backslash [-e, 0]$. Finally, an Airy asymptotic in the neighborhood of $-e$ is deduced.
© 2001 Academic Press
*Key Words:* Stirling numbers of the second kind; generating functions; Plancherel Rotach asymptotics.

## 1. INTRODUCTION AND SUMMARY

This paper contains asymptotic expansions for the horizontal generating function of the Stirling numbers of the second kind $S(n,k)$, which are defined by the following double generating function (see [3, p. 50]):

$$\exp\{z(e^u-1)\} =: 1 + \sum_{1 \leqslant k \leqslant n < \infty} S(n,k) \frac{u^n}{n!} z^k, \qquad z, u \in \mathbb{C}. \tag{1.1}$$

The horizontal generating functions $P_n(z)$ are the coefficients of the power series

$$\exp\{z(e^u-1)\} =: 1 + \sum_{n=1}^{\infty} \frac{P_n(z)}{n!} u^n, \qquad z, u \in \mathbb{C},$$

that gives

$$P_n(z) = \sum_{k=1}^{n} S(n,k)\, z^k = \frac{n!}{2\pi i} \int_{\gamma_0} \frac{\exp\{z(e^t-1)\}}{t^{n+1}}\, dt, \tag{1.2}$$



0021-9045/01 $35.00
Copyright © 2001 by Academic Press
All rights of reproduction in any form reserved.

with $\gamma_0$ a simple closed curve with positive orientation encircling 0. In contrast to the vertical generating function and to the corresponding functions of the Stirling numbers of the first kind, whose particular sums can be computed exactly (see [3, pp. 206, 212]), there is no comparative result for the function $P_n$. Therefore, it is interesting to at least deduce some asymptotic results. Concerning asymptotic characteristics only the case $P_n(1)$, the so-called Bell number, has been investigated so far (see [1], [2, pp. 102–108], [3, pp. 296–297], [6]). Moreover, there is one result respecting the zeros of $P_n$. These are simple, real, and not greater than 0 (see [3, p. 271]). In this work, we deduce two asymptotic expansions for $P_n$ with the help of the saddle point method, which requires that the saddle point and the parameter $n$ are independent of each other. This leads to the function

$$\begin{aligned} Q_n(z) &:= P_n(nz) \\ &= \frac{n!}{2\pi i} \int_{\gamma_0} e^{-n(\ln t - z(e^t - 1))} \frac{dt}{t}, \qquad n \in \mathbb{N}, \quad z \in \mathbb{C}. \end{aligned} \tag{1.3}$$

In accordance with asymptotic results for the classic orthogonal polynomials, see for example the Hermite polynomials [8, p. 201], we obtain the following asymptotics of the Plancherel–Rotach-type:

(i) With $\phi \in (0, \pi)$ there is the oscillating asymptotics

$$Q_n\left(-\frac{\sin\phi}{\phi} e^{\phi\cot\phi}\right) = k_n(\phi)\left(\sin\left(n\left(\pi - \phi + \frac{\sin^2\phi}{\phi}\right) + \eta(\phi)\right) + \mathcal{O}\left(\frac{1}{n}\right)\right),$$

with $k_n(\phi) > 0$ and $\eta(\phi)$ bounded by $\frac{\pi}{2}$ and $\pi$ (see Theorem 3.1).

(ii) With $z \in \mathbb{C} \setminus [-e, 0]$ and $w \in \mathscr{A}$, $zwe^w = 1$, $\mathscr{A} := \{w \in \mathbb{C} \setminus \{0\}: w > -1 \text{ or } w = a + ib,\ b \in (-\pi, \pi) \setminus \{0\},\ a > -b\cot b\}$ it holds that

$$Q_n(z) = \frac{n!}{\sqrt{2\pi n}} \frac{1}{w^n} \exp\left\{\frac{n}{w}(1 - e^{-w})\right\}(1 + w)^{-1/2}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right),$$

where the $\mathcal{O}$-term holds uniformly on every compact subset of $\mathbb{C} \setminus [-e, 0]$ (see Theorems 3.2 and 3.3).

In addition, we investigate the turning point $-e$, which occurs in the interval in (i) by tending $\phi \to 0$. Thereby, we get an Airy-asymptotics, which is called a strong asymptotics, as well as the above mentioned asymptotics.

In addition to this work, the position of the zeros of $Q_n$ is investigated in [4].

## 2. AUXILIARY RESULTS

### 2.1. *Technical Lemmas*

LEMMA 2.1. (i) *Let* $\varphi$ *be in* $(0, \frac{\pi}{2})$; *then* $1 < \frac{\varphi}{\sin \varphi} < \frac{\pi}{2}$.

(ii) *Let* $\varphi$ *be in* $(0, \pi)$ *and* $\alpha$ *in* $[0, \pi - \varphi]$; *then* $\cos \alpha \frac{\varphi}{\sin \varphi} > -1$.

*Proof.* Both (i) and (ii) can be easily checked. ∎

LEMMA 2.2. *Let* $a$, $b$ *be in* $\mathbb{R}$, $a < b$, $f: [a, b] \to \mathbb{R}$, $f \in C^2[a, b]$ *with* $f(a) \geqslant f(b)$ *and* $f''(x) < 0$ *for all* $x$ *in* $(a, b)$; *then for all* $x$ *in* $(a, b)$ $f(x)$ *is greater than* $f(b)$.

*Proof.* It is $f''(x) < 0$ for all $x$ in $(a, b)$; i.e. $f$ is strictly concave on $[a, b]$. If we assume that there is an $x_0$ in $(a, b)$ with $f(x_0) \leqslant f(b)$, there is also a $\lambda$ in $(0, 1)$ satisfying:

$$f(b) \geqslant f(x_0) = f(\lambda a + (1-\lambda)\, b) > \lambda f(a) + (1-\lambda)\, f(b) \geqslant f(b).$$

This is a contradiction and thus the lemma is proved. ∎

### 2.2. *The Solution of the Saddle Point Equation*

To apply the saddle point method to the integral (1.2) it is required (see [7, p. 127]) that the derivative of $p(t) = \ln t$ has a simple zero on the curve $\gamma_0$. Because this is not possible, the function $Q_n(z) = P_n(nz)$ is introduced. By (1.3) this leads to

$$p(t) = \ln t - z(e^t - 1) \qquad \text{and} \qquad p'(t) = \frac{1}{t} - ze^t; \tag{2.1}$$

the logarithm will be defined below. To determine $t$ with $p'(t) = 0$, which is equivalent to solving the equation $zte^t = 1$ with $z \in \mathbb{C} \backslash \{0\}$, we refer to [5], where a similar problem (namely to solve $z(t-1)\, e^t = 1$) is discussed in detail. The obtained solutions and characteristics can be transferred easily and so the following results hold:

LEMMA 2.3. *Define*

$$\mathscr{A} := \{w \in \mathbb{C} \backslash \{0\} : w > -1 \text{ or } w = a + ib,\ b \in (-\pi, \pi) \backslash \{0\},\ a > -b \cot b\},$$

$$\Gamma_+ := \partial\mathscr{A} \cap \{w \in \mathbb{C} : \Im(w) > 0\},$$

$$\Gamma_- := \partial\mathscr{A} \cap \{w \in \mathbb{C} : \Im(w) < 0\},$$

$$\Gamma := \partial\mathscr{A} \backslash \{0\} = \Gamma_+ \cup \Gamma_- \cup \{-1\} \qquad \text{and} \qquad \Psi : \bar{\mathscr{A}} \backslash \{0\} \to \mathbb{C},$$

$$\Psi(w) := \frac{1}{w} e^{-w}, \qquad \text{then:}$$

(i) *The equation* $x = \Psi(w)$, $x \in (-e, 0)$, *has exactly two solutions* $w = a \pm ib$, $w \in \bar{\mathscr{A}}$, *which satisfy*

(a) $a > -1$, $b \in (0, \pi)$, $a = -b \cot b$, $a + ib \in \Gamma_+$, $a - ib \in \Gamma_-$,

(b) $x = h(b) := -\frac{\sin b}{b} e^{b \cot b}$,

(c) $\lim_{b \to 0} h(b) = -e$, $\lim_{b \to \pi} h(b) = 0$, $h(b)$ *is increasing strictly*, $b \in (0, \pi)$,

(d) *with* $x \in (-e, 0)$ $a = a(x)$ *is increasing strictly*, $a((-e, 0)) = (-1, \infty)$, *and* $b = b(x)$ *is increasing strictly*, $b((-e, 0)) = (0, \pi)$,

(e) $w \in \Gamma_+$, $w = a + ib$, *or* $w \in \Gamma_-$, $w = a - ib$, *satisfies* $|w|^2 = \frac{b^2}{\sin^2 b}$.

(ii) $\Psi$ *maps* $\mathscr{A}$ *conformally onto* $\mathbb{C} \backslash [-e, 0]$.

(iii) $\Psi$ *maps both* $\Gamma_+$ *and* $\Gamma_-$ *one-one onto* $(-e, 0)$.

*Proof.* See [5, pp. 346–350]. ▮

Due to these results, an inverse function $\Phi$: $\mathbb{C} \backslash \{-e, 0\} \to \mathscr{A} \cup \Gamma_+$ of $\Psi$ can be defined (see Fig. 1):

$$\Phi(z) := \begin{cases} w,\ w \in \mathscr{A},\ zwe^w = 1, & z \in \mathbb{C} \backslash [-e, 0] \\ w,\ w \in \Gamma_+,\ zwe^w = 1, & z \in (-e, 0). \end{cases} \tag{2.2}$$

The function $\Phi$ is analytic on $\mathbb{C} \backslash [-e, 0]$ and maps $\mathbb{C} \backslash [-e, 0]$ conformally onto $\mathscr{A}$. Especially, it holds for $\Phi((-\infty, -e)) = (-1, 0)$ and $\Phi((0, \infty)) = (0, \infty)$. Further, $\Phi$ solves the saddle point equation:

Lemma 2.4. *For* $z \in \mathbb{C} \backslash \{-e, 0\}$ $p'(t)$ *has a simple zero at* $w = \Phi(z)$.

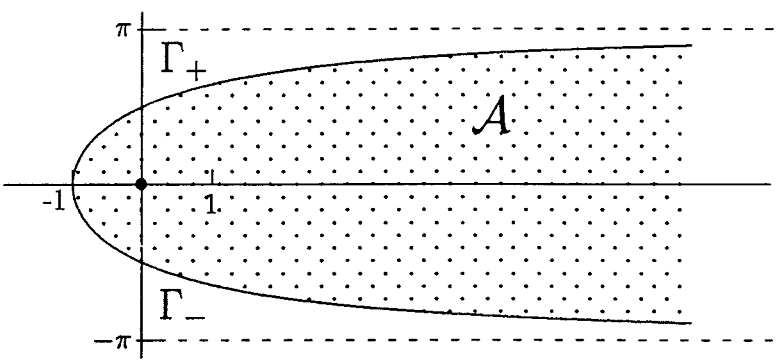


**FIG. 1.** The domain $\mathscr{A}$.

*Proof.* Because of (2.2), $p'(w)=0$, so we have to show $p''(w)\neq 0$:

$$p''(w)=-\frac{1}{w^2}-ze^w=-\frac{1}{w}\left(\frac{1}{w}+1\right)\neq 0, \qquad \text{because } -1\notin \mathscr{A}\cup\Gamma_+. \quad \blacksquare$$

For $z\in(-e,0)$ we introduce a parametrization (compare Lemma 2.3(i)(a), (b))

$$z=x(\phi):=-\frac{\sin(\phi)}{\phi}\,e^{\phi\cot\phi}, \qquad \phi\in(0,\pi), \tag{2.3}$$

so that $w$ is given as:

$$w=w(\phi):=-\phi\cot\phi+i\phi=-\frac{\phi}{\sin\phi}\,e^{-i\phi}=\frac{\phi}{\sin\phi}\,e^{i(\pi-\phi)}. \tag{2.4}$$

The next problem is to determine a curve $\gamma_0$ having $w$ as an interior point and satisfying the condition (see [7, p. 127]) that the real part of $p(t)-p(w)$ is positive for all $t\in\gamma_0\backslash\{w\}$. We will prove that $\gamma_0$ may be chosen as a circle for $z$ in the cut plane $\mathbb{C}\backslash[-e,0]$ and as a semi-circle for $z\in(-e,0)$.

## 3. PLANCHEREL-ROTACH ASYMPTOTICS

### 3.1. *The Oscillating Asymptotics*

For $z\in(-e,0)$, from (1.3) we get the representation

$$\begin{aligned} Q_n(z)&=\frac{n!}{2\pi i}\int_{\gamma_0} e^{-n(\ln t-z(e^t-1))}\,\frac{dt}{t}\\ &=\frac{n!}{\pi}\,\mathfrak{I}\left\{\int_{\gamma_0^+} e^{-n(\ln t-z(e^t-1))}\,\frac{dt}{t}\right\}, \end{aligned} \tag{3.1}$$

with $\gamma_0^+$ the upper half of the circle with radius $|w|$ and $\ln t=\ln|t|+i\,\mathrm{ph}(t)$, $\mathrm{ph}(t)\in[0,\pi]$. If we want to apply the saddle point method to (3.1), we have to verify the real part condition, i.e., $\mathfrak{R}\{p(t)-p(w)\}$ is greater than 0 for all $t\in\gamma_0^+$. By using (2.3) and (2.4) with $r(\phi)=\frac{\phi}{\sin\phi}$, $t=t(\psi):=r(\phi)\,e^{i\psi}$, $\psi\in[0,\pi]$, $w(\phi)=t(\pi-\phi)$, and $R(\psi):=\mathfrak{R}\{p(t(\psi))-p(t(\pi-\phi))\}$ we will prove that $R(\psi)$ is greater than 0 for all $\psi\in[0,\pi]\backslash\{\pi-\phi\}$.

(a) Computation of $R(\psi)$ and $R'(\psi)=\frac{dR(\psi)}{d\psi}$:

$$\begin{aligned}R(\psi)&=\Re\{p(t(\psi))-p(t(\pi-\phi))\}\\&=\Re\{\ln(r(\phi)\,e^{i\psi})-x(\phi)\,e^{r(\phi)\,e^{i\psi}}-\ln(r(\phi)\,e^{i(\pi-\phi)})+x(\phi)\,e^{r(\phi)\,e^{i(\pi-\phi)}}\}\\&=-x(\phi)\,e^{r(\phi)\cos\psi}\cos(r(\phi)\sin\psi)-\frac{1}{r(\phi)}\cos(\phi).\end{aligned}\tag{3.2}$$

That gives:

$$\begin{aligned}R'(\psi)&=-x(\phi)\,e^{r(\phi)\cos\psi}r(\phi)(-\sin\psi)\cos(r(\phi)\sin\psi)\\&\quad-x(\phi)\,e^{r(\phi)\cos\psi}(-\sin(r(\phi)\sin\psi))\,r(\phi)\cos\psi\\&=x(\phi)\,e^{r(\phi)\cos\psi}r(\phi)\sin(\psi+r(\phi)\sin\psi).\end{aligned}\tag{3.3}$$

With $f(\psi):=\psi+r(\phi)\sin\psi$ and $g(\psi):=x(\phi)\,e^{r(\phi)\cos\psi}r(\phi)$ we have:

$$\begin{aligned}&R'(\psi)=g(\psi)\sin(f(\psi)),\\&g(\psi)<0\qquad\text{for all}\quad\psi\in[0,\pi]\qquad\text{and}\\&R'(\psi)=0\qquad\text{if and only if}\quad f(\psi)=k\pi,\quad k\in\mathbb{N}_0.\end{aligned}\tag{3.4}$$

(b) Proof that $R(\psi)$ is greater than 0 for all $\psi$ in $[0,\pi-\phi)$: Because $f(0)=0$, $f(\pi-\phi)=\pi$, and $f'(\psi)=1+\frac{\phi}{\sin\phi}\cos\psi>0$ for $\psi\in(0,\pi-\phi)$ (cf. Lemma 2.1(ii)), by using (3.4) it follows that $R'(\psi)<0$ for $\psi\in(0,\pi-\phi)$. Since $R(\pi-\phi)=0$, the allegation is proved.

(c) Proof that $R(\psi)$ is greater than 0 for all $\psi$ in $(\pi-\phi,\pi]$:

*Case* 1. $\phi\leqslant\frac{\pi}{2}$. It is sufficient to show that $f(\psi)\in(\pi,2\pi)$, for $\psi\in(\pi-\phi,\pi)$, then from (3.4) it follows that $R'(\psi)>0$ for $\psi\in(\pi-\phi,\pi)$. First, by Lemma 2.1(i) it holds that $f(\psi)=\psi+r(\phi)\sin\psi<\pi+\frac{\pi}{2}<2\pi$. On the other hand it holds that $f(\pi-\phi)=f(\pi)=\pi$, $f'(\psi)=1+\cos\psi\,\frac{\phi}{\sin\phi}$ and $f''(\psi)=-\sin\psi\,\frac{\phi}{\sin\phi}<0$ for $\psi\in(\pi-\phi,\pi)$. And thus Lemma 2.2 gives $f(\psi)>\pi$.

*Case* 2. $\phi>\frac{\pi}{2}$. Because $R(\pi)=x(\phi)\,e^{r(\phi)(-1)}\cos(0)-\frac{1}{r(\phi)}\cos(\phi)>0$ and $R(\pi-\phi)=0$, it is sufficient to show that

$$R(\psi_0)>0,\qquad\text{for all}\quad\psi_0\in\mathcal{N},\quad\mathcal{N}:=\{\psi\in(\pi-\phi,\pi):R'(\psi)=0\}.$$

Since $f(\psi_0)>0$ and by (3.4), it follows that for $\psi_0\in\mathcal{N}$ $f(\psi_0)\in\{k\pi\colon k\in\mathbb{N}\}$ if and only if there is a $k_0\in\mathbb{N}$ with $\psi_0+r(\phi)\sin\psi_0=k_0\pi$, i.e. $r(\phi)\sin\psi_0=k_0\pi-\psi_0$.

α. $\psi_0 \in (\pi-\phi, \phi)$, by (3.2) it follows that

$$\begin{aligned} R(\psi_0) &= -x(\phi)\, e^{r(\phi)\cos\psi_0} \cos(r(\phi)\sin\psi_0) + x(\phi)\, e^{r(\phi)\cos(\pi-\phi)} \cos\phi \\ &= x(\phi)\, e^{r(\phi)\cos(\pi-\phi)} \cos\phi \\ &\quad - x(\phi)\, e^{r(\phi)\cos\psi_0} \begin{cases} \cos\psi_0, & k_0 = 2m,\ m\in\mathbb{N} \\ -\cos\psi_0, & k_0 = 2m-1,\ m\in\mathbb{N} \end{cases} \\ &= x(\phi) \left( e^{r(\phi)\cos(\pi-\phi)} \cos\phi - e^{r(\phi)\cos\psi_0} \begin{Bmatrix} \cos\psi_0 \\ -\cos\psi_0 \end{Bmatrix} \right) > 0, \end{aligned}$$

because $\cos(\pi-\phi) = -\cos\phi > |\cos\psi_0| \geqslant 0$, $\psi_0 \in (\pi-\phi, \phi)$, $\phi > \frac{\pi}{2}$.

β. $\psi_0 \in [\phi, \pi)$. This case does not exist, because $\mathcal{N} \cap [\phi, \pi)$ is empty, which is proved as follows:

First, it holds that $f(\phi) = \phi + r(\phi)\sin\phi = 2\phi > \pi$, $f(\pi) = \pi$ and $f''(\psi) = -r(\phi)\sin\psi < 0$ for $\psi \in (\phi, \pi)$. Then Lemma 2.2 with $a=\phi$ and $b=\pi$ gives:

$$f(\psi) > f(\pi) = \pi, \qquad \text{for all} \quad \psi \in [\phi, \pi).$$

Second, for $\psi \in [\phi, \pi)$ it follows that

$$f(\psi) = \psi + \frac{\sin\psi}{\sin\phi}\phi < \pi + 1\pi = 2\pi.$$

That means that $f(\psi) \in (\pi, 2\pi)$ for $\psi \in [\phi, \pi)$ and it follows by (3.4) that $R'(\psi) \neq 0$. ▮

Thus the real part condition is accomplished and the saddle point method may be applied.

THEOREM 3.1. *Let $x(\phi)$ be defined by (2.3). Then for $\phi \in (0, \pi)$, as $n \to \infty$,*

$$Q_n(x(\phi)) = k_n(\phi) \left( \sin\left( n \left( \pi - \phi + \frac{\sin^2\phi}{\phi} \right) + \eta(\phi) \right) + \mathcal{O}\left( \frac{1}{n} \right) \right),$$

*with arccos: $[-1, 1] \to [0, \pi]$, $\eta$: $(0, \pi) \to (\frac{\pi}{2}, \pi)$, $k_n(\phi)$: $(0, \pi) \to (0, \infty)$ and*

$$\begin{aligned} k_n(\phi) &:= \frac{n!}{\sqrt{\pi n}} \sqrt{2}\, e^{-n(\ln(\phi/\sin\phi) + x(\phi) + ((\sin\phi\cos\phi)/\phi))} \\ &\quad \times \left( \left( \frac{\phi}{\sin\phi} - \cos\phi \right)^2 + \sin^2\phi \right)^{-1/4} \\ \eta(\phi) &:= \frac{\pi}{2} + \frac{1}{2} \arccos\left( \frac{1 - \phi\cot(\phi)}{((1-\phi\cot(\phi))^2 + \phi^2)^{1/2}} \right). \end{aligned}$$

*Proof.* According to [7, p. 127] with (2.4), (3.1), and Lemma 2.4 we obtain

$$Q_n(x(\phi)) = \frac{n!}{\pi} \Im \left\{ \int_{\gamma_0^\pm} e^{-n(\ln t - x(\phi)(e^t - 1))} \frac{dt}{t} \right\}$$

$$= \frac{n!}{\pi} \Im \left\{ 2e^{-np(w(\phi))} \frac{1}{\sqrt{n}} \left( \Gamma\left(\frac{1}{2}\right) a_0 + \mathcal{O}\left(\frac{1}{n}\right) \right) \right\},$$

with $a_0 = (w(\phi)\sqrt{2p''(w(\phi))})^{-1}$ and $\omega_0 := \mathrm{ph}(p''(w(\phi)))$ satisfying $|\omega_0 + 2\omega| \leqslant \frac{\pi}{2}$, where $\omega$ is the limiting value of $\mathrm{ph}(t - w(\phi))$ as $t \to w(\phi)$ along $(w(\phi), -|w(\phi)|)$, which means the part of $\gamma_0^+$ between $-|w(\phi)|$ and $w(\phi)$.

(a) Computation of $p(w(\phi))$ gives:

$$p(w(\phi)) = p(t(\pi - \phi)) = \ln t(\pi - \phi) - x(\phi)(e^{t(\pi - \phi)} - 1)$$

$$= \ln r(\phi) + i(\pi - \phi) + x(\phi)$$

$$+ \frac{\sin \phi}{\phi} e^{\phi \cot \phi} e^{(\phi/\sin \phi)(\cos(\pi - \phi) + i \sin(\pi - \phi))}$$

$$= \ln \frac{\phi}{\sin \phi} + x(\phi) + \frac{\sin \phi \cos \phi}{\phi} + i \left( \pi - \phi + \frac{\sin^2 \phi}{\phi} \right). \tag{3.5}$$

(b) With $t \to w(\phi)$ along $(w(\phi), -|w(\phi)|)$, $\omega$ is given by:

$$\omega = \lim_{t \to w(\phi)} \mathrm{ph}(t - w(\phi)) = \lim_{\psi \to (\pi - \phi),\, \psi > (\pi - \phi)} \mathrm{ph}(r(\phi)(e^{i\psi} - e^{i(\pi - \phi)}))$$

$$= \pi - \phi + \lim_{\varepsilon \to 0,\, \varepsilon > 0} \mathrm{ph}(e^{i\varepsilon} - 1) = \tfrac{3}{2}\pi - \phi. \tag{3.6}$$

(c) Since $x(\phi)\, w(\phi)\, e^{w(\phi)} = 1$ and due to Lemma 2.4, $\omega_0$ is computed as follows,

$$\omega_0 = \mathrm{ph}(p''(w(\phi))) = \mathrm{ph}\left( -\frac{1}{w(\phi)^2} (1 + w(\phi)) \right)$$

$$= \mathrm{ph}\left( \frac{e^{-i\pi} e^{-2i(\pi - \phi)}}{(r(\phi))^2} \right) + \mathrm{ph}(1 + w(\phi)) = -3\pi + 2\phi + \mathrm{ph}(1 + w(\phi)),$$

and since $\Re(w(\phi)) > -1$, the condition $|\omega_0 + 2\omega| \leqslant \frac{\pi}{2}$ is satisfied. With $w(\phi) = r(\phi)\, e^{i(\pi - \phi)}$ and $\arccos : [-1, 1] \to [0, \pi]$, $\omega_0$ holds further:

$$\omega_0 = -3\pi + 2\phi + \arccos\left( \frac{1 - \phi \cot(\phi)}{((1 - \phi \cot(\phi))^2 + \phi^2)^{1/2}} \right). \tag{3.7}$$

(d) Computation of $a_0$ gives:

$$
\begin{aligned}
a_0 &= (w(\phi)\sqrt{2p''(w(\phi))})^{-1} \\
&= \frac{1}{\sqrt{2}} \frac{e^{-i(\pi-\phi)}}{r(\phi)} e^{-i(\omega_0/2)} \left| \frac{1}{r(\phi)\, e^{i(\pi-\phi)}} \left( \frac{1}{r(\phi)\, e^{i(\pi-\phi)}} + 1 \right) \right|^{-1/2} \\
&= \frac{1}{\sqrt{2}} e^{-i(\pi-\phi+(\omega_0/2))} \, |e^{-i(\pi-\phi)} + r(\phi)|^{-1/2} \\
&= \frac{1}{\sqrt{2}} e^{-i(\pi-\phi+(\omega_0/2))} \left[ \left( \frac{\phi}{\sin\phi} - \cos\phi \right)^2 + \sin^2\phi \right]^{-1/4}.
\end{aligned} \tag{3.8}
$$

Altogether, from (3.5), (3.7), and (3.8) we obtain:

$$
\begin{aligned}
Q_n(x(\phi)) &= \frac{n!}{\sqrt{\pi n}} \Im \Bigg\{ 2e^{-n(\ln(\phi/\sin\phi)+x(\phi)+((\sin\phi\cos\phi)/\phi)+i(\pi-\phi+(\sin^2\phi/\phi)))} \\
&\quad \times \left( \frac{1}{\sqrt{2}} e^{-i(\pi-\phi+(\omega_0/2))} \left[ \left( \frac{\phi}{\sin\phi} - \cos\phi \right)^2 + \sin^2\phi \right]^{-1/4} \right. \\
&\quad \left. + \mathcal{O}(n^{-1}) \right) \Bigg\} \\
&= \frac{n!}{\sqrt{\pi n}} \sqrt{2}\, e^{-n(\ln(\phi/\sin\phi)+x(\phi)+((\sin\phi\cos\phi)/\phi))} \\
&\quad \times \left[ \left( \frac{\phi}{\sin\phi} - \cos\phi \right)^2 + \sin^2\phi \right]^{-1/4} \\
&\quad \times \left( \sin\left( \phi - \pi - \frac{\omega_0}{2} - n\left( \pi - \phi + \frac{\sin^2\phi}{\phi} \right) \right) + \mathcal{O}(n^{-1}) \right) \\
&= \frac{n!}{\sqrt{\pi n}} \sqrt{2}\, e^{-n(\ln(\phi/\sin\phi)+x(\phi)+((\sin\phi\cos\phi)/\phi))} \\
&\quad \times \left[ \left( \frac{\phi}{\sin\phi} - \cos\phi \right)^2 + \sin^2\phi \right]^{-1/4} \left( \sin\left( n\left( \pi - \phi + \frac{\sin^2\phi}{\phi} \right) \right. \right. \\
&\quad \left. \left. + \frac{\pi}{2} + \frac{1}{2} \arccos\left( \frac{1-\phi\cot(\phi)}{((1-\phi\cot(\phi))^2+\phi^2)^{1/2}} \right) \right) + \mathcal{O}(n^{-1}) \right). \quad \blacksquare
\end{aligned}
$$

3.2. *The Asymptotics on the Cut Plane*

For $z \in \mathbb{C} \backslash [-e, 0]$ we will apply the saddle point method to (1.3). Therefore we will choose $\gamma_0$ as a circle with radius $|w|$. The logarithm is defined by $\ln t = \ln |t| + i\,\mathrm{ph}(t)$ with $\mathrm{ph}(t) \in [0, 2\pi]$ for $z \in \mathbb{C}\setminus[-e, \infty)$ and $\mathrm{ph}(t) \in [-\pi, \pi]$ for $z \in (0, \infty)$.

First, we have to prove the real part condition again; i.e., $\Re\{p(t) - p(w)\}$ is greater than 0 for all $t \in \gamma_0\setminus\{w\}$. With $w = re^{i\alpha}$, $r > 0$, $\alpha \in [0, 2\pi)$, $w \in \mathscr{A}$, $r < r_{\max} := \frac{\pi-\alpha}{\sin\alpha}$ for $\alpha \in (0, \pi) \cup (\pi, 2\pi)$ ($r < r_{\max} := 1$ for $\alpha = \pi$) and $t = t(\psi) = re^{i\psi}$, $\psi \in [0, 2\pi]$, for $\alpha \in (0, 2\pi)$ ($[-\pi, \pi]$ for $\alpha = 0$), we have to show that $R(\psi) := \Re\{p(t(\psi)) - p(w)\}$ is greater than 0 for all $\psi \neq \alpha$.

By (2.1) and $zwe^w = 1$ computation of $R(\psi)$ gives:

$$R(\psi) = \Re\{-ze^{t(\psi)} + ze^w\} = \Re\left\{\frac{1}{r}\,e^{-i\alpha}(1 - e^{r(e^{i\psi} - e^{i\alpha})})\right\}$$

$$= \frac{1}{r}\cos\alpha - \frac{1}{r}\,e^{r(\cos\psi - \cos\alpha)}\cos(\alpha - r(\sin\psi - \sin\alpha)). \qquad (3.9)$$

If $\alpha = 0$ the allegation follows directly from (3.9). For $\alpha \neq 0$ we define

$$f(r, \psi) := e^{r(\cos\psi - \cos\alpha)}\cos(\alpha - r(\sin\psi - \sin\alpha)),$$

$$\mathscr{G} := \{(r, \psi) \in \mathbb{R}^2 : 0 < r < r_{\max},\ \psi \in (0, 2\pi + \varepsilon)\}$$

$$\text{with}\quad 0 < \varepsilon < \min\{\alpha, 2\pi - \alpha\},$$

$$\mathscr{Q} := \{(r_0, \psi_0) \in \overline{\mathscr{G}} : f(r_0, \psi_0) \geqslant f(r, \psi) \text{ for all } (r, \psi) \in \overline{\mathscr{G}}\}, \qquad \text{and}$$

$$\mathscr{M} := \{(r, \psi) \in \overline{\mathscr{G}} : r = 0 \text{ or } r = r_{\max},\ \psi \in \{\alpha, 2\pi - \alpha\} \text{ or } \psi = \alpha\}.$$

Hence, it is sufficient to show that $f(r, \psi) = \cos\alpha$ for all $(r, \psi) \in \mathscr{M}$ and $\mathscr{Q} \subset \mathscr{M}$. The first condition can be easily checked, and thus we only have to prove $\mathscr{Q} \subset \mathscr{M}$. For $(r_*, \psi_*) \in \mathscr{Q}$ it follows directly that $f(r_*, \psi_*)$ is not less than $\cos\alpha$. To show that $(r_*, \psi_*) \in \mathscr{M}$, we investigate in (a), (b), (c), and (d) possible maximums on the edge of $\mathscr{G}$ and in (e) in the interior of $\mathscr{G}$.

(a) $\mathbf{r_* = 0.}$ $(r_*, \psi_*) \in \mathscr{M}$ follows immediately.

(b) $\mathbf{r_* = r_{max}.}$ To show that $\psi_* \in \{\alpha, 2\pi - \alpha\}$ we define $g(\psi) := f(r_{\max}, \psi)$, then

$$g(\psi) = \begin{cases} e^{((\pi-\alpha)/\sin\alpha)(\cos\psi - \cos\alpha)}\cos\left(\alpha - \dfrac{\pi-\alpha}{\sin\alpha}(\sin\psi - \sin\alpha)\right), & \alpha \in (0, 2\pi)\setminus\{\pi\} \\ e^{\cos\psi - \cos\pi}\cos(\pi - (\sin\psi - \sin\pi)), & \alpha = \pi \end{cases}$$

$$= \begin{cases} -e^{(\alpha-\pi)\cot\alpha}e^{(\pi-\alpha/\sin\alpha)\cos\psi}\cos\left(\dfrac{\pi-\alpha}{\sin\alpha}\sin\psi\right), & \alpha \in (0, 2\pi)\setminus\{\pi\} \\ -e^{1+\cos\psi}\cos(\sin\psi), & \alpha = \pi, \end{cases} \qquad (3.10)$$

so we have to prove $g(\psi) < \cos\alpha$ for all $\psi \in [0, 2\pi + \varepsilon]\setminus\{\alpha, 2\pi - \alpha\}$.

(b1) $\boldsymbol{\alpha \in (0, \pi), \psi \in [0, \pi]\backslash\{\alpha\}}$. With $\alpha =: \pi - \phi$, $\phi \in (0, \pi)$, it follows from Section 3.1, especially (3.2), that $g(\psi) < \cos\alpha$.

(b2) $\boldsymbol{\alpha \in (0,\pi), \psi \in (\pi, 2\pi]\backslash\{2\pi - \alpha\}}$. By using $\varphi \in [0, \pi)\backslash\{\alpha\}$ with $\psi = 2\pi - \varphi$, it follows from (3.10) that $g(\psi) = g(\varphi)$ and hence by (b1), $g(\psi) < \cos\alpha$ for all $\psi \in (\pi, 2\pi]\backslash\{2\pi - \alpha\}$.

(b3) $\boldsymbol{\alpha \in (0, \pi), \psi \in (2\pi, 2\pi + \varepsilon]}$. By (b1) and the $2\pi$-periodicity of $g$ it holds that $g(\psi) < \cos\alpha$.

(b4) $\boldsymbol{\alpha = \pi}$. By (3.10) we have to show that $h(\psi) := e^{\cos\psi}\cos(\sin\psi) > \frac{1}{e}$ for all $\psi \in [0, 2\pi + \varepsilon]\backslash\{\pi\}$. Since $h'(\psi) = -e^{\cos\psi}\sin(\psi + \sin\psi)$, the function $h$ has an absolute minimum in $[0, 2\pi + \varepsilon]$ at $\psi = \pi$ and by $h(\pi) = e^{-1}$ the prove is completed.

(b5) $\boldsymbol{\alpha \in (\pi, 2\pi)}$. With $\alpha = 2\pi - \beta$, $\beta \in (0, \pi)$, it follows from (3.10) that

$$g(\psi) = -e^{(\beta - \pi)\cot\beta} e^{((\pi - \beta)/\sin\beta)\cos\psi} \cos\left(\frac{\pi - \beta}{\sin\beta}\sin\psi\right).$$

Hence, we can deduce immediately from (b1)–(b3) that $g(\psi) < \cos\alpha$ for all $\psi \in [0, 2\pi + \varepsilon]\backslash\{\alpha, 2\pi - \alpha\}$.

(c) $\boldsymbol{r_* \in (0, r_{\max}), \psi_* = 0}$. By the $2\pi$-periodicity of $f$ with reference to $\psi$, it follows that $(r_*, 2\pi) \in \mathcal{Q}$ also. This is investigated in (e).

(d) $\boldsymbol{r_* \in (0, r_{\max}), \psi_* = 2\pi + \varepsilon}$. By the $2\pi$-periodicity of $f$ with reference to $\psi$, it follows that also $(r_*, \varepsilon) \in \mathcal{Q}$. This is investigated in (e).

(e) $\boldsymbol{r_* \in (0, r_{\max}), \psi_* \in (0, 2\pi + \varepsilon)}$. We assume: $\psi_* \neq \alpha$. Because $(r_*, \psi_*) \in \mathcal{Q}$, $(r_*, \psi_*) \in \mathcal{G}$, and $\mathcal{G}$ is open, $(r_*, \psi_*)$ must comply with:

$$\frac{\partial f(r, \psi)}{\partial r} = \frac{\partial f(r, \psi)}{\partial \psi} = 0, \qquad (r, \psi) = (r_*, \psi_*). \tag{3.11}$$

Computation of $\partial f/\partial r$ gives:

$$\begin{aligned}\frac{\partial f(r, \psi)}{\partial r} &= e^{r(\cos\psi - \cos\alpha)}((\cos\psi - \cos\alpha)\cos(\alpha - r(\sin\psi - \sin\alpha)) \\ &\quad - \sin(\alpha - r(\sin\psi - \sin\alpha))(-(\sin\psi - \sin\alpha))) \\ &= -2e^{r(\cos\psi - \cos\alpha)}\left(\sin\left(\frac{\psi - \alpha + 2r(\sin\psi - \sin\alpha)}{2}\right)\sin\left(\frac{\psi - \alpha}{2}\right)\right).\end{aligned}$$

That means $f_r(r_*, \psi_*) = 0$ if and only if $\psi_* - \alpha = 2k\pi$, $k \in \mathbb{Z}$ or $\psi_* - \alpha + 2r_*(\sin\psi_* - \sin\alpha) = 2k\pi$, $k \in \mathbb{Z}$.

*Case* 1. $\psi_* - \alpha = 2k\pi$, $k \in \mathbb{Z}$, by $\psi_* = \alpha + 2k\pi$; it follows that

if $k = 0$: $\psi_* = \alpha$, contradiction to the assumption!
if $k < 0$: $\psi_* < 0$, contradiction to $\psi_* \in \mathcal{G}$!
if $k > 0$: $\psi_* = 2k\pi + \alpha > 2\pi + \varepsilon$, contradiction to $\psi_* \in \mathcal{G}$!

*Case* 2. $\psi_* - \alpha + 2r_*(\sin\psi_* - \sin\alpha) = 2k\pi$, $k \in \mathbb{Z}$; that means:

$$r_*(\sin\psi_* - \sin\alpha) = k\pi + \frac{\alpha - \psi_*}{2}. \tag{3.12}$$

By

$$\begin{aligned}\frac{\partial f(r,\psi)}{\partial\psi} &= e^{r(\cos\psi - \cos\alpha)}((-r\sin\psi)\cos(\alpha - r(\sin\psi - \sin\alpha))\\ &\quad - \sin(\alpha - r(\sin\psi - \sin\alpha))(-r\cos\psi))\\ &= -re^{r(\cos\psi - \cos\alpha)}(\sin(\psi - \alpha + r(\sin\psi - \sin\alpha))),\end{aligned}$$

and (3.11), it follows that $\psi_* - \alpha + r_*(\sin\psi_* - \sin\alpha) = m\pi$, $m \in \mathbb{Z}$, and hence by (3.12), $\psi_* - \alpha + k\pi + \frac{\alpha - \psi_*}{2} = m\pi$. That means $\psi_* = \alpha + 2(m-k)\pi$ and we can deduce:

if $m = k$: $\psi_* = \alpha$, a contradiction to the assumption!
if $m < k$: $\psi_* < 0$, a contradiction to $\psi_* \in \mathcal{G}$!
if $m > k$: $\psi_* > 2\pi + \varepsilon$, a contradiction to $\psi_* \in \mathcal{G}$!

Hence, we have shown that neither Case 1 nor Case 2 can occur and our assumption must be wrong. ∎

Thus, the real part condition is accomplished and the saddle point method may be applied.

THEOREM 3.2. *Let $z$ be in $\mathbb{C} \setminus [-e, 0]$ and $\Phi(z) = w \in \mathcal{A}$, $zwe^w = 1$. Then, as $n \to \infty$:*

$$Q_n(z) = \frac{n!}{\sqrt{2\pi n}} \frac{1}{w^n} \exp\left\{\frac{n}{w}(1 - e^{-w})\right\}(1+w)^{-1/2}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right).$$

*Proof.* According to [7, p. 127] and (3.1) it follows that

$$\begin{aligned}Q_n(z) &= \frac{n!}{2\pi i}\int_{\gamma_0} e^{-n(\ln t - z(e^t - 1))}\frac{dt}{t}\\ &= \frac{n!}{\pi i} e^{-np(w)}\frac{1}{\sqrt{n}}\left(\Gamma\left(\frac{1}{2}\right)a_0 + \mathcal{O}\left(\frac{1}{n}\right)\right),\end{aligned}$$

with $a_0 = (w\sqrt{2p''(w)})^{-1}$ and $\omega_0 := \mathrm{ph}(p''(w))$ satisfying $|\omega_0 + 2\omega| \leqslant \frac{\pi}{2}$, where $\omega$ is the limiting value of $\mathrm{ph}(t-w)$ as $t \to w$ along the part of $\gamma_0$ between $w$ and the endpoint of $\gamma_0$.

(a) By $zwe^w = 1$ computation of $p(w)$ gives:

$$p(w) = \ln w - z(e^w - 1) = \ln w - \frac{1}{w} + \frac{e^{-w}}{w}. \tag{3.13}$$

(b) With $w = |w|\, e^{i\alpha}$ and $t = |w|\, e^{i\psi}$, $\omega$ is given by:

$$\omega = \lim_{t \to w} \mathrm{ph}(t-w) = \lim_{\psi \to \alpha,\, \psi > \alpha} \mathrm{ph}(|w|\, e^{i\psi} - |w|\, e^{i\alpha})$$

$$= \lim_{\varepsilon \to 0,\, \varepsilon > 0} \mathrm{ph}(|w|\, e^{i\alpha}(e^{i\varepsilon} - 1)) = \alpha + \frac{\pi}{2}. \tag{3.14}$$

(c) By $zwe^w = 1$, $w = |w|\, e^{i\alpha}$, and Lemma 2.4, $\omega_0$ is computed as follows

$$\omega_0 = \mathrm{ph}(p''(w)) = \mathrm{ph}\left( -\frac{1}{w^2}(1+w) \right)$$

$$= \mathrm{ph}\left( e^{-i\pi} \frac{1}{|w|^2} e^{-2i\alpha}(1+w) \right) = -\pi - 2\alpha + \mathrm{ph}(1+w), \tag{3.15}$$

and since $\Re(w) > -1$, the condition $|\omega_0 + 2\omega| \leqslant \frac{\pi}{2}$ is satisfied.

(d) Computation of $a_0$ gives

$$a_0 = (w\sqrt{2p''(\omega)})^{-1} = \frac{1}{\sqrt{2}} \frac{e^{-i\alpha}}{|w|} \left| -\frac{1}{w^2}(1+w) \right|^{-1/2} e^{-(1/2)\, i\omega_0}$$

$$= \frac{1}{\sqrt{2}} |1+w|^{-1/2}\, e^{i(\pi/2)} e^{-(i/2)\,\mathrm{ph}(1+w)} = \frac{i}{\sqrt{2}}(1+w)^{-1/2}, \tag{3.16}$$

with $\ln(1+w) = \ln|1+w| + i\,\mathrm{ph}(1+w)$, $|\mathrm{ph}(1+w)| \leqslant \frac{\pi}{2}$.

From (3.13), (3.15), and (3.16) we now obtain:

$$Q_n(z) = \frac{n!}{\pi i} \exp\left\{ -n\left( \ln w - \frac{1}{w} + \frac{e^{-w}}{w} \right) \right\}$$

$$\times \left( \sqrt{\frac{\pi}{n}} \frac{i}{\sqrt{2}} (1+w)^{-1/2} + \mathcal{O}(n^{-3/2}) \right)$$

$$= \frac{n!}{\sqrt{2\pi n}} \frac{1}{w^n} \exp\left\{ \frac{n}{w}(1 - e^{-w}) \right\} ((1+w)^{-1/2} + \mathcal{O}(n^{-1})). \quad \blacksquare$$

The result of Theorem 3.2 can be strengthened: the asymptotics is valid uniformly in every compact set $K$ lying in $\mathbb{C}\backslash[-e, 0]$.

THEOREM 3.3. *Let $K$ be a compact subset of $\mathbb{C}\backslash[-e, 0]$ and*

$$G_n(z) := \frac{n!}{\sqrt{2\pi n}} \frac{1}{w^n} \exp\left\{\frac{n}{w}(1-e^{-w})\right\}(1+w)^{-1/2},$$

$$z \in \mathbb{C}\backslash[-e, 0], \quad w = \Phi(z).$$

*Then $Q_n(z)/G_n(z)$ converges uniformly to 1 on $K$.*

*Proof.* Since the saddle point method is a special application of Laplace's method, the following proof is based on the proof of this method (see [7, pp. 121–125]) with the difference that the function $p(t)$ depends on a parameter $z$ besides:

$$p(z, t) := \ln t - z(e^t - 1), \qquad t \in \mathbb{C}\backslash\{0\}, \quad z \in K.$$

The branch of the logarithm must be chosen in a suitable way, so that $p$ is analytic in the sets appearing below (therefore $K$ has to be split into two compact sets, if necessary). Further, we must mention that if $s$ is odd, the coefficients used below $a_s$ disappear when the Laplace method is replaced by the saddle point method. This happens because a different $\omega_0$ (see (3.15)) has to be chosen (compare with the variable $v$ introduced below with $\mathrm{ph}(v) = \omega_0$).

(a) Because $z \in K$, $K$ compact, there is an $M_1 > 0$ with $|w|^{-1} = |\Phi(z)|^{-1} \leqslant M_1$ for all $z \in K$. Further, the functions $p$ and $q(t) = t^{-1}$ have the following power series representations in a neighborhood of $w$:

$$p(z, t) = p(z, w) + \sum_{s=0}^{\infty} p_s(z)(t-w)^{s+\mu}, \qquad \mu = 2,$$

$$p_s(z) = \frac{1}{(s+2)!} \frac{\partial^{(s+2)} p}{\partial t^{(s+2)}}(z, w), \qquad \frac{\partial^s p}{\partial t^s}(z, w) = \frac{(-1)^{s-1}(s-1)!}{w^s} - \frac{1}{w},$$

$$q(t) = \sum_{s=0}^{\infty} q_s (t-w)^{s+\lambda-1}, \qquad q_s = \frac{(-1)^s}{w^{s+1}}, \quad \lambda = 1. \tag{3.17}$$

Hence, we obtain the following estimations for $|p_s(z)|$ and $|p_0(z)|$, which hold for all $z \in K$:

$$|p_s(z)| \leqslant \frac{1}{(s+2)!}\left((s+1)!\, M_1^{s+2} + M_1\right) \leqslant (M_1+1)^{s+2} =: M_2^{s+2}, \tag{3.18}$$

$$|p_0(z)| = \left|\frac{1}{2}\left(-\frac{1}{w^2} - \frac{1}{w}\right)\right| \geqslant M_3 > 0, \tag{3.19}$$

$$\frac{1}{|p_0(z)|} \leqslant M_4, \qquad M_4 := \max\left\{1, \frac{1}{M_3}\right\}. \tag{3.20}$$

(b) We set $u = g(z, t) := \sqrt{p(z, t) - p(z, w)}$ with $\ln u = \ln |u| + i\,\mathrm{ph}(u)$, $\mathrm{ph}(u) \in [-\pi, \pi)$; then (3.17) gives:

$$\begin{aligned} u &= \left(\sum_{s=0}^{\infty} p_s(z)(t-w)^{s+2}\right)^{1/2} \\ &= \left(p_0(z)(t-w)^2\left(1 + \sum_{s=1}^{\infty} \frac{p_s(z)}{p_0(z)}(t-w)^s\right)\right)^{1/2} \\ &= \sqrt{p_0(z)}\,(t-w)\left(1 + \sum_{k=1}^{\infty} \binom{1/2}{k}\left(\sum_{s=1}^{\infty} \frac{p_s(z)}{p_0(z)}(t-w)^s\right)^k\right) \\ &= \sqrt{p_0(z)}\,(t-w) \\ &\quad \times \left(1 + \sum_{m=1}^{\infty} (t-w)^m \left(\sum_{k=1}^{m} \binom{1/2}{k} \sum_{\substack{v_1 + \cdots + v_k = m \\ v_i \geqslant 1}} \prod_{i=1}^{k} \frac{p_{v_i}(z)}{p_0(z)}\right)\right). \end{aligned}$$

(c) Let $g(z, t) = \sum_{m=0}^{\infty} g_m(z)(t-w)^{m+1}$; then there is an $M_6$ greater than 0 with $|g_m(z)| \leqslant M_2 M_6^m$ for all $z \in K$.

*Proof.* The case $m = 0$ follows immediately from $g_0(z) = \sqrt{p_0(z)}$ and (3.18), so we consider $m \in \mathbb{N}$, $1 \leqslant k \leqslant m$. By (3.18) and (3.20) it follows that

$$\left|\prod_{i=1}^{k} \frac{p_{v_i}(z)}{p_0(z)}\right| \leqslant \prod_{i=1}^{k} M_2^{v_i+2} M_4 \leqslant M_4^k M_2^{m+2k} \leqslant (M_4 M_2^3)^m =: M_5^m.$$

Since $|\binom{1/2}{k}| \leqslant 1$ and $\sum_{v_1 + \cdots + v_k = m,\, v_i \geqslant 1} 1 = \binom{m-1}{k-1}$, we obtain:

$$\begin{aligned} |g_m(z)| &= \left|\sqrt{p_0(z)} \sum_{k=1}^{m} \binom{1/2}{k} \sum_{\substack{v_1 + \cdots + v_k = m \\ v_i \geqslant 1}} \prod_{i=1}^{k} \frac{p_{v_i}(z)}{p_0(z)}\right| \\ &\leqslant M_2 \sum_{k=1}^{m} \binom{m-1}{k-1} M_5^m \\ &= M_2 M_5^m \sum_{k=0}^{m-1} \binom{m-1}{k} \leqslant M_2 (2M_5)^m =: M_2 M_6^m. \end{aligned} \tag{3.21}$$

Especially, we can conclude that $g(z, t)$ converges in the circle $\{t: |t-w| < M_6^{-1}\}$ for all $z \in K$.

(d) We define $n_0 := \max\{3, 1 + 3M_2(M_3)^{-1/2}\}$; then for all $z$ in $K$ $g(z, t)$ maps $\{t: |t-w| < (n_0 M_6)^{-1}\}$ conformally onto a domain $U$ with $0 \in U$.

*Proof.* Let $z$ be in $K$, $|t_i - w| < (n_0 M_6)^{-1}$, $i = 1, 2$, $t_1 \neq t_2$; then by (3.19) and (c) it follows that $|g(z, t_1) - g(z, t_2)|$

$$= \left| g_0(z)(t_1 - t_2) + (t_1 - t_2) \sum_{m=2}^{\infty} g_{m-1}(z) \frac{(t_1 - w)^m - (t_2 - w)^m}{(t_1 - w) - (t_2 - w)} \right|$$

$$= \left| g_0(z)(t_1 - t_2) + (t_1 - t_2) \sum_{m=2}^{\infty} g_{m-1}(z) \sum_{v=0}^{m-1} (t_1 - w)^{m-1-v} (t_2 - w)^v \right|$$

$$\geqslant |t_1 - t_2| \left( \sqrt{M_3} - \sum_{m=2}^{\infty} M_2 M_6^{m-1} m \left( \frac{1}{n_0 M_6} \right)^{m-1} \right)$$

$$= |t_1 - t_2| \left( \sqrt{M_3} - M_2 \frac{2n_0 - 1}{(n_0 - 1)^2} \right)$$

$$> |t_1 - t_2| \left( \sqrt{M_3} - 3M_2 \frac{1}{n_0 - 1} \right) \geqslant 0,$$

$$\text{because} \quad n_0 \geqslant 1 + \frac{3M_2}{\sqrt{M_3}}.$$

(e) Since $\Phi(K)$ is compact, we can choose $0 < R \leqslant (2n_0 M_6)^{-1}$ so that $\{t : |t-w| \leqslant R\}$ is contained in $\mathscr{A} \backslash \{0\}$. Further, we define:

$D_R(z) := \{t: t = w + Re^{i\psi}, \psi \in [0, 2\pi]\}$,
$L(z) := g(z, D_R(z)) = \{u: u = g(z, t), t \in D_R(z)\}$, and
$r(z) := \mathrm{dist}(L(z), 0) = \min\{|u|: u \in L(z)\}$.

Since $r$ is continuous on $K$ and positive, there is an $r_0$ greater than 0 with $r_0 = \min\{r(z) : z \in K\}$. So the set $\{u : |u| < r_0\}$ lies in the interior of $L(z)$ for all $z$ in $K$. Therefore, we can deduce that for all $z$ in $K$ there exists a $\delta(z) > 0$ satisfying $|g(z, we^{i\delta(z)})| = r_0$, $|w(e^{i\delta(z)} - 1)| \leqslant R$, and $|g(z, we^{i\psi})| < r_0$ for all $\psi \in [0, \delta(z))$. Finally, we set $\delta_0 > 0$ with $\delta(z) \geqslant \delta_0$ for all $z$ in $K$.

(f) Now, we define $k(z) := we^{i\delta_0}$ and $\kappa(z) := g(z, k(z))^2$, so we see that $\kappa(z)$ is continuous on $K$ and $\Re(\kappa(z))$ is greater than 0 for all $z$ in $K$. Considering the real part condition, there must be a $\kappa_0$ and a $\kappa_{\max}$ with:

$$\kappa_0 > 0, \qquad \kappa_0 = \min\{\Re(\kappa(z)): z \in K\}, \tag{3.22}$$

$$\kappa_{\max} > 0, \qquad \kappa_{\max} = \max\{|\kappa(z)|: z \in K\}. \tag{3.23}$$

(g) For $z$ in $K$ we get (see [7, p. 123, set $n=2$ and $z=n$]),

$$\int_w^{k(z)} e^{-np(z,t)} q(t)\,dt = e^{-np(z,w)} \int_0^{\kappa(z)} e^{-nv} f(v)\,dv,$$

with $v=u^2=(g(z,t))^2$, $f(v)=q(t)\frac{dt}{dv}=q(t)/\frac{\partial p(z,t)}{\partial t}$ and

$$f(v)=\sum_{s=0}^{1} a_s v^{(s-1)/2} + \sqrt{v}\, f_2(v), \qquad f_2(v)=\mathcal{O}(1), \quad v\to 0.$$

So we deduce:

$$\int_0^{\kappa(z)} e^{-nv} f(v)\,dv = \sum_{s=0}^{1} \Gamma\left(\frac{s+1}{2}\right) \frac{a_s}{n^{(s+1)/2}} - \varepsilon_{2,1}(n,z) + \varepsilon_{2,2}(n,z).$$

The error terms can be estimated uniformly:

(g1) As mentioned above, the coefficient $a_1$ does not have to be taken into account, so it follows that $\varepsilon_{2,1}(n,z)=\Gamma(\frac{1}{2}, \kappa(z)\,n)\,(q_0/\sqrt{2p_0})\,1/\sqrt{n}$. Further, the incomplete gamma function holds:

$$\Gamma(\tfrac{1}{2}, \kappa(z)\,n) = \int_{\kappa(z)\,n}^{\infty} e^{-t} t^{-1/2}\,dt = e^{-\kappa(z)\,n} \int_{\kappa(z)\,n}^{\infty} e^{-(t-\kappa(z)\,n)} t^{-1/2}\,dt$$

$$= e^{-\kappa(z)\,n} \int_0^{\infty} e^{-x}(x+\kappa(z)\,n)^{-1/2}\,dx.$$

By (3.22) we obtain $|\Gamma(\frac{1}{2}, \kappa(z)\,n)| \leqslant e^{-\kappa_0 n}(\kappa_0)^{-1/2}$, and by (3.20), $|\varepsilon_{2,1}(n,z)| \leqslant e^{-\kappa_0 n}(\kappa_0)^{-1/2} M_1 (\frac{M_4}{2})^{1/2}\, 1/\sqrt{n}$; that means

$$\varepsilon_{2,1}(n,z) = \frac{1}{\sqrt{n}}\, \mathcal{O}(e^{-\kappa_0 n}) \tag{3.24}$$

and the $\mathcal{O}$-term holds uniformly for all $z$ in $K$.

(g2) Because $f_2(v)=v^{-1/2}(f(v)-\sum_{s=0}^{1} a_s v^{(s-1)/2}) = \sum_{s=2}^{\infty} a_s v^{(s/2)-1}$ and $f_2(v)=\mathcal{O}(1)$, $v\to 0$, there must be an $M_7$ greater than 0 with $|f_2(v)|\leqslant M_7$ for all $z$ in $K$, $|v|$ less than $r_0^2$, and it follows that

$$|\varepsilon_{2,2}(n,z)| = \left| \int_0^{\kappa(z)} e^{-nv} v^{1/2} f_2(v)\,dv \right|$$

$$= \left| \kappa(z) \frac{1}{n^{3/2}} \int_0^{n} e^{-\kappa(z)\,x} (\kappa(z))^{1/2}\, x^{1/2} f_2\left(\frac{\kappa(z)\,x}{n}\right) dx \right|$$

$$\leqslant (\kappa_{\max})^{3/2}\, n^{-3/2} \int_0^{\infty} e^{-\kappa_0 x} x^{1/2} M_7\,dx =: M_8 n^{-3/2};$$

that means

$$\varepsilon_{2,2}(n, z) = \mathcal{O}(n^{-3/2}), \tag{3.25}$$

and the $\mathcal{O}$-term holds uniformly for all $z$ in $K$.

(g3) The final error to calculate is the value of the integral along the arc of the semicircle from $we^{i\delta_0}$ to $-w$. The function $\tilde{g}(z, \psi) := \Re(p(z, we^{i\psi}) - p(z, w))$ is continuous on $K \times [\delta_0, \pi]$ and we deduce from the real part condition that there is an $M_9$ greater than 0 with $\tilde{g}(z, \psi)$ not less than $M_9$ for all $(z, \psi)$ in $K \times [\delta_0, \pi]$. So we get

$$\begin{aligned} \left| \int_{k(z)}^{-w} e^{-np(z,t)} q(t)\, dt \right| &= \left| e^{-np(z,w)} \int_{k(z)}^{-w} e^{-n(p(z,t) - p(z,w))} q(t)\, dt \right| \\ &\leqslant |e^{-np(z,w)}|\, e^{-nM_9} M_1 \left| \int_{k(z)}^{-w} 1\, dt \right| \\ &\leqslant |e^{-np(z,w)}|\, e^{-nM_9} M_1 M_{10}, \end{aligned}$$

with $M_{10}$ greater than 0. Altogether, we obtain

$$\int_{k(z)}^{-w} e^{-np(z,t)} q(t)\, dt = e^{-np(z,w)} \mathcal{O}(e^{-nM_9}), \tag{3.26}$$

where the $\mathcal{O}$-term holds uniformly for all $z$ in $K$ again.

(h) From Theorem 3.2 and (3.24), (3.25), (3.26), and (3.16) we finally obtain:

$$\begin{aligned} Q_n(z) &= G_n(z)(1 + \mathcal{O}(e^{-\kappa_0 n}) + \mathcal{O}(n^{-1}) + \sqrt{n}\, \mathcal{O}(e^{-nM_9})) \\ &= G_n(z)(1 + \mathcal{O}(n^{-1})). \end{aligned}$$

The $\mathcal{O}$-term holds uniformly for all $z$ in $K$ and so the proof is completed. ∎

### 3.3. *The Airy-asymptotics*

Finally, we give an Airy-asymptotics for $Q_n$ as $z \to -e$:

THEOREM 3.4. *Let $z_n = -e(1 - (6n^2)^{-1/3} s)$, $s \in \mathbb{C}$, then, as $n \to \infty$,*

$$\begin{aligned} Q_n(z_n) = \frac{n!}{\pi} (-1)^n \exp\left\{ (e-1) \left( n - \left(\frac{n}{6}\right)^{1/3} s \right) \right\} \left(\frac{6}{n}\right)^{1/3} \\ \times (A(s) + \mathcal{O}(n^{-1/3})), \end{aligned}$$

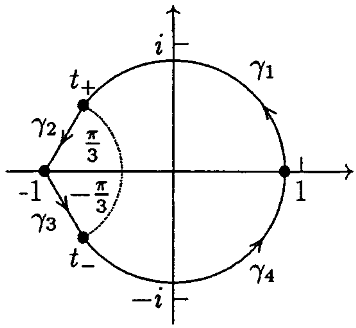


**FIG. 2.** The path of integration for the Airy-asymptotics.

*where $A(s)$ is Airy's function. $A(s)$ is an entire function given by*

$$A(s) = \frac{1}{2\pi i} \int_L \exp\left\{\frac{1}{3} t^3 - st\right\} dt, \qquad s \in \mathbb{C},$$

*where L is any contour which begins at infinity in the sector* $-\frac{\pi}{2} < \mathrm{ph}(t) < -\frac{\pi}{6}$ *and ends at infinity in the sector* $\frac{\pi}{6} < \mathrm{ph}(t) < \frac{\pi}{2}$; *see* [9, p. 90] *and* [8, p. 377]. *Further, the $\mathcal{O}$-term holds uniformly for s in K, K compact.*

*Proof.* The result can be proved in a similar way as in [8, pp. 232–235] for Laguerre polynomials. The main difference is the path of integration $\gamma_0$, which should be chosen here in the following way (see Fig. 2):

Let $\theta$ be in $(0, \frac{1}{12})$, $t_+ = t_+(n) := -1 + (6n^{-1})^{1/3} n^\theta e^{\pi i/3}$, $\alpha_+ := \mathrm{ph}(t_+) \in (0, \pi)$, $t_- = t_-(n) := -1 + (6n^{-1})^{1/3} n^\theta e^{-\pi i/3}$, $\alpha_- := \mathrm{ph}(t_-) \in (\pi, 2\pi)$ and $r_n := |t_+|$. Further we define

$$\begin{aligned}
&\gamma_0 := \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4,\\
&\gamma_1 := \{t\colon |t| = r_n,\ \mathrm{ph}(t) \in [0, \alpha_+]\},\\
&\gamma_2 := \{t\colon t = -1 - (6n^{-1})^{1/3} e^{\pi i/3} p,\ p \in [-n^\theta, 0]\},\\
&\gamma_3 := \{t\colon t = -1 + (6n^{-1})^{1/3} e^{-\pi i/3} p,\ p \in [0, n^\theta]\}, \text{ and}\\
&\gamma_4 := \{t\colon |t| = r_n,\ \mathrm{ph}(t) \in [\alpha_-, 2\pi]\}.
\end{aligned}$$

The rest of the proof is quite similar to that mentioned above with the only difference that the calculations for $\gamma_2$ and $\gamma_3$ lead to the Airy function. ∎

## REFERENCES

1. F. E. Binet and G. Szekeres, On Borel fields over finite sets, *Ann. of Math. Statist.* **28** (1957), 494–498.
2. N. de Bruijn, “Asymptotic Methods in Analysis,” fourth ed., Dover, New York, 1981.

3. L. Comtet, "Advanced Combinatorics," revised and enlarged edition, Dordrecht, Netherlands, 1974.
4. C. Elbert, Weak asymptotics for the generating polynomials of the Stirling numbers of the second kind, submitted for publication.
5. D. S. Lubinsky and A. Sidi, Strong asymptotics for polynomials biorthogonal to powers of log $x$, *Analysis* **14** (1994), 341–379.
6. L. Moser and M. Wyman, An asymptotic formula for the Bell numbers, *Trans. Roy. Soc. Canada* **49** (1955), 49–54.
7. F. W. J. Olver, "Asymptotics and Special Functions," Academic Press, New York/San Francisco/London, 1974.
8. G. Szegő, "Orthogonal Polynomials," fourth ed., American Mathematical Society, Providence, RI, 1975.
9. R. Wong, "Asymptotic Approximations of Integrals," Academic Press, San Diego, 1989.